Time Translation and Kinematics: A Short Derivation

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This short companion to my paper, *Time Translation and the Taylor Structure of Kinematics*, in *The Physics Teacher*, develops the constant-jerk argument through the translated position equation. The idea is to determine an unknown coefficient by requiring the kinematic relations to retain their form when the time origin changes.

From uniform acceleration to uniform jerk

In elementary kinematics, uniform acceleration gives a velocity that varies linearly and a position that varies quadratically. A natural extension introduces a uniform jerk j_0. Since jerk is the rate of change of acceleration, acceleration now takes the same linear form that velocity takes under uniform acceleration. Extending this one-step shift through the kinematic chain gives the quadratic term in velocity. For position, the pattern suggests the next power of time, but not its coefficient. Writing that coefficient as $1/N$, we obtain

$$\begin{aligned} a(t) &= a_0 + j_0 t, \\ v(t) &= v_0 + a_0 t + \frac{1}{2} j_0 t^2, \\ x(t) &= x_0 + v_0 t + \frac{1}{2} a_0 t^2 + \frac{1}{N} j_0 t^3. \end{aligned} \tag{1}$$

Here x_0, v_0, and a_0 are the position, velocity, and acceleration at the reference time $t = 0$. The remaining task is to determine N.

Translate the time origin

Translate $t \to t + h$, treating h as the new time variable. Conceptually, we pause the clock at time t and begin measuring time again from $h = 0$. Substitution into the position relation gives

$$x(t+h) = x_0 + v_0(t+h) + \frac{1}{2} a_0 (t+h)^2 + \frac{1}{N} j_0 (t+h)^3.$$

Expanding the powers and regrouping in powers of h yields

$$\begin{aligned} x(t+h) = {} & \left(x_0 + v_0 t + \frac{1}{2} a_0 t^2 + \frac{1}{N} j_0 t^3 \right) \\ & + \left(v_0 + a_0 t + \frac{3}{N} j_0 t^2 \right) h \\ & + \frac{1}{2} \left(a_0 + \frac{6}{N} j_0 t \right) h^2 + \frac{1}{N} j_0 h^3. \end{aligned} \tag{2}$$

Match the translated coefficients

The motion is unchanged by restarting the clock. Thus the parenthesized quantities in Eq. (2) must be the position, velocity, and acceleration at time t. The first is already $x(t)$. Comparing the other two with Eq. (1) requires

$$v_0 + a_0 t + \frac{3}{N} j_0 t^2 = v_0 + a_0 t + \frac{1}{2} j_0 t^2,$$
$$a_0 + \frac{6}{N} j_0 t = a_0 + j_0 t.$$

Both comparisons give the same condition:

$$\frac{3}{N} = \frac{1}{2}, \qquad \frac{6}{N} = 1, \qquad \text{so} \qquad N = 6.$$

The translated position relation therefore becomes

$$x(t+h) = x_t + v_t h + \frac{1}{2} a_t h^2 + \frac{1}{6} j_0 h^3. \tag{3}$$

Here $x_t = x(t)$, $v_t = v(t)$, and $a_t = a(t)$. Equation (3) expresses the motion relative to an arbitrary reference instant t, with h measuring the elapsed interval from that instant. Setting $t = 0$ recovers the initial-time position relation; setting $h = 0$ gives the position at the reference instant.

For the extension to higher rates of change and the resulting Taylor series structure, see the published paper, *Time Translation and the Taylor Structure of Kinematics* by William F. Barnes.